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Simulation of stochastic heat conduction processes

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Abstract—This paper considers stochastic three-dimensional non-stationary temperature fields described by stochastic heat conduction equations with random coefficients and by stochastic initial and boundary conditions. A numerical method is suggested allowing one to determine non-stationary and stationary three-dimensional fields of expected values, correlations and dispersions of stochastic non-stationary temperature fields in three-dimensional bodies of complicated shape. The method is based on stochastic mathematical model discretization by the methods of finite differences or elements and on the solution of the Volterra stochastic integral equations. Random functions figuring in the mathematical model are the functions of coordinates and time with limited realization. An example of the use of the method is considered, and comparison is given between the results obtained and exact data.

1. INTRODUCTION

THE TEMPERATURE fields of real objects under real conditions of their functioning are stochastic. The stochastic character of the temperature field depends on the dominating random factors. Often such factors, such as the powers of heat sources and sinks, thermal conductivity coefficients, coefficients of heat transfer from the body surface to the medium, ambient temperature, and the width of the gap between contacting bodies are random. The statistical variation of these factors can be substantial. It results from the tolerances in manufacturing processes, non-controlled random factors and random fluctuations of external parameters that characterize heat transfer with surrounding medium. The consequence is that the temperature at each point of the body and at each time instant represents a random variable.

In those cases when random variation of temperature in a body is insignificant and can be ignored, the temperature field is rather precisely described by the deterministic heat conduction equation. If the random character of the temperature field cannot be neglected or high requirements are imposed on the adequacy of the temperature distribution simulation, then it is necessary to solve the stochastic heat conduction equation.

The most practically important characteristics of the stochastic temperature fields are the fields of its first and second moments, viz. of expectation and dispersion. These will be called the solutions of the stochastic heat conduction equation.

Stochastic temperature fields are described by stochastic heat conduction equations with stochastic boundary and initial conditions. Generally, coefficients in the operators of the equation and boundary conditions are random functions of coordinates and time.

The following methods have been widely used to date for solving stochastic heat conduction equations: perturbation methods [1–3], stochastic Green's function method [4, 5], methods allowing one to obtain equations for expectation and dispersion fields [3, 6–8], stochastic numerical methods of finite elements and finite differences [9–11]. In some cases one succeeds in finding an analytical solution of stochastic heat conduction equations [12–15]. It is assumed in many works that random coefficients in the equation and boundary conditions are white Gaussian noises. For this case, using Ito's stochastic integral [5–7], partial differential equations were obtained for three-dimensional non-stationary expectation fields and stochastic temperature field dispersions. In ref. [8], using the method of time-ordered cumulants, equations were obtained for non-stationary one-dimensional distributions of expected values and for the dispersion of the integral of stochastic temperature. The method of finite elements for the non-stationary stochastic heat conduction equation has been developed for the case of a volumetric source assigned in the form of the white Gaussian noise [9]. The method of finite elements and finite differences has been developed for the stationary stochastic heat conduction equation with random coefficients [10, 11].

In the present paper a numerical method is suggested for determining non-stationary three-dimensional fields of expected values and dispersion of the stochastic temperature field. The coefficients entering into the stochastic heat conduction equation and stochastic boundary conditions are random functions of coordinates and time.

NOMENCLATURE

c heat capacity
$c(x,t,\omega)$ random function
$E\{\cdot\}$ expected value operator
G, ∂G three-dimensional region of a body, boundary of the region
K_*, K correlation matrices of stochastic temperatures at grid nodes
k number of points of time interval decomposition
$k(x,t,\omega)$ stochastic thermal conductivity coefficient
n number of nodes in the grid covering the region
$Q(x,t,\omega)$, $q(x,t,\omega)$ stochastic volumetric and surface heat flux densities
$T(x,\omega)$ stochastic stationary three-dimensional temperature field
$T(x,t,\omega)$ stochastic non-stationary three-dimensional temperature field
$T_a(x,t,\omega)$ stochastic ambient temperature
$T_b(x,t,\omega)$ stochastic temperature distribution at body boundary
$T_0(x,\omega)$ stochastic initial temperature distribution
$\bar{T}_*$, $\bar{T}$ expected value vectors of stochastic temperatures at grid nodes
t time
Δt time step
$x = (x_1, x_2, x_3)$ point in three-dimensional space.

Greek symbols
$\alpha(x,t,\omega)$ stochastic coefficient of heat transfer from body surface to medium
ρ material density
σ_*, σ standard deviations vectors of stochastic temperatures at grid nodes
τ time interval
ω elementary events.

2. STOCHASTIC MATHEMATICAL MODEL

In a general case, the stochastic non-stationary temperature field $T = T(x,t,\omega)$ in the three-dimensional body G of the three-dimensional space $x = (x_1, x_2, x_3)$ with the boundary ∂G is described by the following stochastic heat conduction equation in $(x,t,\omega) \in G \times [0,\tau] \times \Omega$:

$$\rho c \frac{\partial T}{\partial t} + A(x,t,\omega)T = Q(x,t,\omega), \tag{1}$$

with initial condition in $(x,\omega) \in G \times \Omega$:

$$T(x,0,\omega) = T_0(x,\omega), \tag{2}$$

and boundary conditions in $(x,t,\omega) \in \partial G \times (0,\tau) \times \Omega$:

$$T = T_b(x,t,\omega), \tag{3}$$

$$k(x,t,\omega)\frac{\partial T}{\partial n} = q(x,t,\omega), \tag{4}$$

$$k(x,t,\omega)\frac{\partial T}{\partial n} + \alpha(x,t,\omega)[T - T_a(x,t,\omega)] = q(x,t,\omega), \tag{5}$$

which can be prescribed simultaneously on different parts of the boundary ∂G. The stochastic operator $A(x,t,\omega)$ has the form:

$$A(x,t,\omega)T = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[k(x,t,\omega)\frac{\partial T}{\partial x_i}\right] + c(x,t,\omega)T. \tag{6}$$

In equations (1)–(6) the quantity ω denotes elementary events in the space of elementary events Ω from the measurable space $(\Omega, \sigma, \mathscr{P})$ with the probability measure $\mathscr{P}$ prescribed on the σ-algebra.

In practice, all the random functions in the stochastic mathematical model (1)–(5) and operator (6) vary within a range. Therefore, their probability densities for each $x \in G + \partial G$ and $t \in [0,\tau]$ are truncated, i.e. they are prescribed within the ranges of variation of random functions and are equal to zero beyond these intervals. It is assumed that all of the random functions are statistically independent.

The three-dimensional body G may have an arbitrary shape and consist of several dissimilar materials.

3. DISCRETE ANALOG OF THE STOCHASTIC MATHEMATICAL MODEL

Apply the method of finite differences [16] or the method of finite elements [17] to equations (1)–(5) with operator (6) in the region $G + \partial G$ for each $\omega \in \Omega$. Then the stochastic discrete analog of stochastic mathematical model (1)–(6) will be obtained in the form of the matrix system of stochastic ordinary differential equations:

$$P\frac{\mathrm{d}T(t,\omega)}{\mathrm{d}t} + R(t,\omega)T(t,\omega) = F(t,\omega) + S(t,\omega)\Phi(t,\omega),$$

$$PT(0,\omega) = T_0(\omega), \tag{7}$$

where $T(t,\omega) = (T_1(t,\omega) \ldots T_n(t,\omega))^{\mathrm{T}}$ is the stochastic vector of temperatures at the n nodes of the grid covering the region $G + \partial G$; $R(t,\omega)$ and $S(t,\omega)$ are the familiar stochastic $n \times n$ matrices; P is the deterministic $n \times n$ matrix; $F(t,\omega)$, $\Phi(t,\omega)$ and $T_0(\omega)$ are the familiar stochastic n-vectors which are statistically independent of one another and of the matrices $R(t,\omega)$ and $S(t,\omega)$. The matrices $R(t,\omega)$ and $S(t,\omega)$ are statistically dependent.

If the stochastic matrix equation (7) is obtained by the method of finite elements, then the vector $T(t,\omega)$ represents the vector of random coefficients $T_i(t,\omega)$ entering into the linear combination:

$$T(x,t,\omega) = \sum_{n=1}^{n} T_i(t,\omega)\varphi_i(x), \tag{8}$$

where $\varphi_i(x)$ is a system of expansion functions.

The stochastic matrices $R(t,\omega)$ and $S(t,\omega)$ obtained by both methods can be different but they are of band and symmetric type. The deterministic matrix P in the method of finite elements is of band and symmetric type; in the method of finite differences the matrix $P = I$, where I signifies the unit diagonal matrix.

The stochastic matrix differential equation (7) is equivalent to the Volterra second-kind stochastic integral equation for each $\omega \in \Omega$:

$$PT(t,\omega) - PT(0,\omega) + \int_0^t R(\xi,\omega)T(\xi,\omega)\,\mathrm{d}\xi = \int_0^t [F(\xi,\omega)+S(\xi,\omega)\Phi(\xi,\omega)]\,\mathrm{d}\xi. \tag{9}$$

Replace the integrals in equation (9) by quadrature formulae [18]. For this purpose, divide the time interval $[0,t]$ into k intervals by the points $0, t_1, t_2, \ldots, t_k$. Then, for the time instant t_i, $i = 1,2,\ldots,k$ $(t_0 = 0$, $t_k = t)$ the stochastic integral equation (9) will be transformed into a system of stochastic matrix linear algebraic equations:

$$PT_i - PT_0 + \sum_{j=0}^{i} a_jR_jT_j = \sum_{j=0}^{i} a_j(F_j + S_j\Phi_j),$$

$$i = 1,2,\ldots,k, \tag{10}$$

where $T_j = T(t_j,\omega)$, $\Phi_j = \Phi(t_j,\omega)$, $F_j = F(t_j,\omega)$, $S_j = S(t_j,\omega)$, $R_j = R(t_j,\omega)$ are the stochastic vectors and matrices at the time instant t_j; a_j are numerical coefficients in the quadrature formula.

The integrals can be replaced by finite sums with the aid of various formulae, say, the Simpson trapezoidal rule.

For the clarity of presentation assume that the initial temperature distribution is equal to zero, $T_0 = 0$; and that the points t_i, $i = 1,2,\ldots,k$, are distributed uniformly with the step Δt. As a quadrature formula, use will be made of the trapezoidal rule, so that the coefficients in equation (10) are equal to: $a_0 = a_k = \Delta t/2$, $a_1 = a_2 = \ldots = a_{k-1} = \Delta t$.

After simple transformations, the system of stochastic matrix linear equations (10) can be reduced to the following partitioned matrix system of equations in the unknown lump vector $T_*(\omega) = [T_1(\omega)\ldots T_k(\omega)]^{\mathrm{T}}$:

$$R_*(\omega)T_*(\omega) = \frac{\Delta t}{2}[F_*(\omega)+S_*(\omega)\Phi_*(\omega)], \tag{11}$$

where $R_*(\omega)$ is the stochastic partitioned $(kn)\times(kn)$-matrix of the form:

$$R_*(\omega) = \begin{bmatrix} \frac{\Delta t}{2}R_1(\omega)+P & & \\ -2P & \frac{\Delta t}{2}R_2(\omega)+P & \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \\ (-1)^{k-1}2P & (-1)^{k-2}2P & \frac{\Delta t}{2}R_k(\omega)+P \end{bmatrix}. \tag{12}$$

$S_*(\omega)$ is the stochastic partitioned diagonal $(kn)\times(kn)$-matrix along the diagonal of which there are stochastic matrices $S_1(\omega), S_2(\omega),\ldots, S_k(\omega)$; $F_*(\omega) = (F_1(\omega)\ldots F_k(\omega))^{\mathrm{T}}$ and $\Phi_*(\omega) = (\Phi_1(\omega)\ldots\Phi_k(\omega))^{\mathrm{T}}$ are stochastic lump (kn)-vectors; the superscript T means transpose operation.

The $F_*(\omega)$ and $\Phi_*(\omega)$ vectors are statistically independent of one another and of the matrices $R_*(\omega)$ and $S_*(\omega)$.

4. DETERMINATION OF THE NON-STATIONARY STOCHASTIC TEMPERATURE VECTOR

Let us represent the elements $r_{(i)ml}$ $(m,l = 1,2,\ldots,n)$ of the matrix $R_i(\omega)$ in the form of the sum of the expected value of $\bar{r}_{(i)ml}$ and random value of $r^0_{(i)ml}(\omega)$ with $E\{r^0_{(i)ml}(\omega)\} = 0$, i.e. $r_{(i)ml}(\omega) = \bar{r}_{(i)ml}+r^0_{(i)ml}$. From this, a similar equality follows for the matrix $R_i(\omega)$, i.e. $R_i(\omega) = \bar{R}_i + R^0_i(\omega)$ with $E\{R^0_i(\omega)\} = 0$. Then the stochastic matrix $R_*(\omega)$ can be presented as:

$$R_*(\omega) = \bar{R}_* + \frac{\Delta t}{2}R^0_*(\omega),$$

where $\bar{R}_*$ is the deterministic partitioned $(kn)\times(kn)$-matrix of the form:

$$R_*(\omega) = \begin{bmatrix} \frac{\Delta t}{2}\bar{R}_1P & & \\ -2P & \frac{\Delta t}{2}\bar{R}_2+P & \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \\ (-1)^{k-1}2P & (-1)^{k-2}2P\cdots & \frac{\Delta t}{2}\bar{R}_k+P \end{bmatrix} \tag{13}$$

$R^0_*(\omega)$ is the stochastic partitioned diagonal $(kn)\times(kn)$-matrix along the diagonal of which there are stochastic matrices $R^0_1(\omega), R^0_2(\omega),\ldots, R^0_k(\omega)$.

The stochastic system of equations (11) has the stochastic vector $T_*(\omega)$ as its solution if the stochastic matrix $R_*(\omega)$ for each $\omega \in \Omega$ has its reciprocal. The matrix $R_*(\omega)$ can be inverted if for all $\omega \in \Omega$ the following inequality holds $(\Delta t/2)\|\bar{R}_*^{-1} \cdot R_*^0(\omega)\| < 1$. Under this condition, the stochastic matrix $R_*(\omega)$ can be presented by an infinite converging matrix series [19]:

$$R_*^{-1}(\omega) = \left\{\bar{R}_*\left[I + \frac{\Delta t}{2} H_*(\omega)\right]\right\}^{-1}$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[\frac{\Delta t}{2} H_*(\omega)\right]^i \bar{R}_*^{-1}, \quad (14)$$

where $H_*(\omega) = \bar{R}_*^{-1} R_*^0(\omega)$; I is the unit diagonal $(kn) \times (kn)$-matrix.

The matrix norm $\|C(\omega)\|$ of the stochastic matrix $C(\omega)$, having limited realizations, will be understood to represent the largest deterministic matrix norm for all $\omega \in \Omega$. Since the realizations of the random elements $c_{ij}(\omega)$ of the stochastic matrix $C(\omega)$ are limited and obey the truncated distribution laws in the intervals $[c'_{ij}, c''_{ij}]$, it is evident that $\max_{\omega \in \Omega}\|C(\omega)\|$ is attained when each random element $c_{ij}(\omega)$ takes on one of the values c''_{ij} or c''_{ij}.

Given an expression for the reciprocal stochastic matrix $R_*^{-1}(\omega)$, the system of equations (11) will yield the stochastic vector of temperatures:

$$T_*(\omega) = \frac{\Delta t}{2} \sum_{i=0}^{\infty} (-1)^i \left[\frac{\Delta t}{2} H_*(\omega)\right]^i \bar{R}_*^{-1}$$

$$\times [F_*(\omega) + S_*(\omega)\Phi_*(\omega)]. \quad (15)$$

5. NON-STATIONARY MOMENTS OF THE STOCHASTIC VECTOR OF TEMPERATURES

The stochastic vector of temperatures $T_*(\omega)$, which is determined by expression (15), allows one to determine its moments, viz.: the vector of the expected value $\bar{T}_* = E\{T_*(\omega)\}$, correlation matrix $K_* = E[T_*(\omega) \times T_*^{\mathrm{T}}(\omega)]$, covariance matrix $C_* = E[T_*^0(\omega) T_*^{0^{\mathrm{T}}}(\omega)] = K_* - \bar{T}_* \bar{T}_*^{\mathrm{T}}$, $T_*^0(\omega) = T_*(\omega) - \bar{T}_*$, the vector of dispersions D_* equal to the diagonal of the covariance matrix C_* and the vector of standard deviations $\sigma_* = \sqrt{D_*}$.

Let us apply the operator of the expected value to the stochastic vector $T_*(\omega)$ and to the product $T_*(\omega) T_*^{\mathrm{T}}(\omega)$. Resorting to the statistical independence of the vectors F_* and Φ_* of one another and of the matrices $R_*^0(\omega)$ and $S_*(\omega)$, we obtain:

the vector of the expected value:

$$\bar{T}_* = \frac{\Delta t}{2} \sum_{i=0}^{\infty} (-1)^i E$$

$$\times \left\{\left[\frac{\Delta t}{2} H_*(\omega)^i \bar{R}_*^{-1} [\bar{F}_* + S_*(\omega)\, \bar{\Phi}_*]\right]\right\}; \quad (16)$$

the correlation matrix:

$$K_* = \frac{(\Delta t)^2}{4} \sum_{i,j=0}^{\infty} (-1)^{i+j} E$$

$$\times \left\{\left[\frac{\Delta t}{2} H_*(\omega)\right]^i \bar{R}_*^{-1} Z_*(\omega) (\bar{R}_*^{-1})^{\mathrm{T}}\right.$$

$$\left.\times \left[\frac{\Delta t}{2} H_*^{\mathrm{T}}(\omega)\right]^j\right\}, \quad (17)$$

where $z_*(\omega)$ is the stochastic $(kn) \times (kn)$-matrix equal to:

$$Z_*(\omega) = K_{\mathrm{FF}} + K_{\mathrm{F\Phi}} S_*^{\mathrm{T}}(\omega) + S_*(\omega) K_{\mathrm{F\Phi}}^{\mathrm{T}}$$

$$+ S_*(\omega) K_{\Phi\Phi} S_*^{\mathrm{T}}(\omega). \quad (18)$$

$\bar{F}_* = E[F_*(\omega)]$ and $\bar{\Phi}_* = E[\Phi_*(\omega)]$ are the vectors of expectations; $K_{\mathrm{FF}} = E[F_*(\omega) F_*^{\mathrm{T}}(\omega)]$, $K_{\mathrm{F\Phi}} = E[F_*(\omega) \Phi_*^{\mathrm{T}}(\omega)]$, $K_{\Phi\Phi} = E[\Phi_*(\omega) \Phi_*^{\mathrm{T}}(\omega)]$ are the familiar correlation $(kn) \times (kn)$-matrices.

Naturally, for practical calculations of the vector $\bar{T}_*$ and matrix K_* some first terms in series (16) and in a double series (17) are taken. Series (16) and (17) converge rapidly and, as a rule, with an accuracy sufficient for practice it is possible to restrict ourselves to the terms involving the matrix $(\Delta t/2) H_*(\omega)$ with the degrees not higher than two, and, in some cases, not higher than four.

Expressions (16) and (17) can be easily programmed by applying the technique of operation with partitioned matrices and taking into account that the matrix $\bar{R}_*$ has a simple structure, the matrix $R_*^0(\omega)$ is of partitioned-diagonal type, and the matrices $R_i^0(\omega)$ $(i = 1, 2, \ldots, k)$ have a band structure and are symmetric. In accordance with the structure of the matrix $\bar{R}_*$ (13), the reciprocal matrix $\bar{R}_*^{-1}$ is also partitioned under triangular matrix with the matrices-blocks R_{*ij} having the dimension $n \times n$. It can be easily shown that the matrices-blocks R_{*ij}, $i \geqslant j$, $i = 1, 2, \ldots, k$, can be calculated from the following recurrent formulae:

$$R_{*ij} = \left(\frac{\Delta t}{2} \bar{R}_i + P\right)^{-1}, \quad i = j,$$

$$R_{*ij} = 2R_{*ij} P\{R_{*i-1,j} - R_{*i-2,j} + \cdots + (-1)^{i-j+1} R_{*jj}\},$$

$$i \geqslant j.$$

The number of terms in the latter formula is equal to $i - j$. The calculation of the reciprocal matrix $\bar{R}_*^{-1}$ from the above-given formulae is convenient for programming. As compared with direct computation, it requires much smaller expenditure of memory and machine time.

6. STATIONARY MOMENTS OF THE STOCHASTIC VECTOR OF TEMPERATURES

The stationary stochastic temperature field $T(x, \omega)$ in the three-dimensional region G of the three-dimensional space $x = (x_1, x_2, x_3)$ with the boundary ∂G is described by the stationary stochastic equation of heat conduction in $(x,\omega)\in G\times\Omega$:

$$A(x,\omega)T(x,\omega) = Q(x,\omega), \tag{19}$$

with the time-independent stochastic operator $A(x,\omega)$ of form (6) and with time-independent boundary conditions of form (3)–(5) determined in $(x,\omega)\in\partial G\times\Omega$.

The random functions of the stationary mathematical model for all $x\varepsilon G+\partial G$ satisfy the conditions listed in section 2.

Replacing equation (19) with the boundary conditions by their discrete analog by the method of finite differences or by the method of finite elements, we shall obtain the stochastic matrix system of linear equations for determining the vector of stochastic temperatures $T(\omega) = [T_1(\omega)\dots T_n(\omega)]^{\mathrm{T}}$ at n nodes of the grid:

$$R(\omega)T(\omega) = F(\omega)+S(\omega)\Phi(\omega). \tag{20}$$

Representing the stochastic matrix $R(\omega)$ as a sum of its expected value $\bar{R}$ and stochastic matrix $R^0(\omega)$ with $E[R^0(\omega)] = 0$, we shall expand (just as it was done in Section 4) the reciprocal stochastic matrix $R^{-1}(\omega) = \{\bar{R}[I+\bar{R}^{-1}R^0(\omega)]\}^{-1}$ into a converging matrix series and obtain from equations (20) the stochastic vector:

$$T(\omega) = \sum_{i=0}^{\infty}(-1)^iH^i(\omega)\bar{R}^{-1}[F(\omega)+S(\omega)\Phi(\omega)], \tag{21}$$

provided that $\|H(\omega)\| < 1$, where $H(\omega) = \bar{R}^{-1}R^0(\omega)$. Applying the expected value operator to the vector $T(\omega)$ and to the product $T(\omega)T^{\mathrm{T}}(\omega)$, we shall obtain the stationary moments of the stochastic temperature vector, viz. the n-vector of the expected values:

$$\bar{T} = \sum_{i=0}^{\infty}(-1)^iE\{H^i(\omega)\bar{R}^{-1}[\bar{F}+S(\omega)\bar{\Phi}]\}; \tag{22}$$

the correlation $n\times n$-matrix:

$$K = \sum_{i,j=0}^{\infty}(-1)^{i+j}E\{H^i(\omega)\bar{R}^{-1}Z(\omega) \times(\bar{R}^{-1})^{\mathrm{T}}[H^{\mathrm{T}}(\omega)]^j\}, \tag{23}$$

where $Z(\omega)$ is the stochastic $n\times n$-matrix of form (18).

For practical calculations from formulae (22) and (23), it is usually sufficient to limit ourselves to the series terms involving the matrices $H(\omega)$ with the degrees not higher than two or four.

7. NUMERICAL EXAMPLE

As an example demonstrating the application of the proposed method, consider a one-dimensional plate of thickness $2d$ with the stochastic heat conduction coefficient $k(x,\omega)$. Deterministic temperatures are assigned on the planes of the plate. The initial temperature distribution is equate to zero. The stochastic non-stationary temperature field of the plate $T(x,t,\omega)$ is described by the heat conduction equation for $x\in[-d,d]$ and $t>0$:

$$\rho c\frac{\partial T(x,t,\omega)}{\partial t} = \frac{\partial}{\partial x}\left[k(x,t,\omega)\frac{\partial T(x,t,\omega)}{\partial x}\right],$$

with deterministic boundary conditions for $t>0$:

$$T(-d,t,\omega) = T_1,\quad T(d,t,\omega) = T_2,$$

with initial condition:

$$T(x,0,\omega) = 0.$$

The stochastic thermal conductivity coefficient represents a random function of the coordinate x and is equal to:

$$k(x,\omega) = a(x)+\psi(\omega)b(x),$$

where $\psi(\omega)$ is a random quantity with $E[\psi(\omega)] = 0$ which obeys a certain truncated law of distribution within the range of its variation $[-\psi_{\mathrm{m}},\psi_{\mathrm{m}}]$; $a = k_{\mathrm{m}}+k_{\mathrm{d}}x/2d$, $b = x^2/d^2-1$; k_1, k_2 are the deterministic thermal conductivity coefficients on the boundaries of the plate $x=-d$ and $x=d$, respectively; $k_{\mathrm{m}} = (k_1+k_2)/2$, $k_{\mathrm{d}} = k_2-k_1$.

A similar problem, was considered in [2] where a stochastic analysis of the contact heat conduction problem was performed.

Under the conditions of the example considered, the following initial numerical data were adopted: $d = 0.02$ m, $k_1/\rho c = 3\times10^{-5}$ m^2 s^{-1}, $k_2/\rho c = 5\times10^{-6}$ m^2 s^{-1}, $T_1 = 150$°C, $T_2 = 25$°C. The random quantity $\psi(\omega)$ obeys the uniform distribution law with the probability density $f(\psi) = 1/2\psi_{\mathrm{m}}$ within the interval $[-\psi_{\mathrm{m}},\psi_{\mathrm{m}}]$ and $f(\psi) = 0$ outside the interval $[-\psi_{\mathrm{m}},\psi_{\mathrm{m}}]$, where $\psi_{\mathrm{m}}/\rho c = 9.7\times10^{-6}$ m^2 s^{-1}. The standard deviation of the random quantity $\psi(\omega)$ is equal to $\sigma_\psi/\rho c = 5.6\times10^{-6}$ m^2 s^{-1} and comprises 58% of the ψ_{m} value.

To obtain the discrete analog of the mathematical model, the method of finite differences was used. The plate $[-d,d]$ was covered by a uniform grid with the nodes $x_i = ih$, $i = 0,1,\dots,20$ and step $h = 2d/20$. The node $x_0 = 0$ is located on the plane $x=-d$, the node $x_{20} = 20$ is situated on the plane $x = d$ and the node $x_{10} = 10h$ is located at the centre of the plate at $x = 0$. The time from zero to 180 s was uniformly split into $k = 5$ intervals with the step $\Delta t = 36$ s.

Calculations of non-stationary and stationary distribution of the expected value $\bar{T} = \bar{T}(x,t)$ and of the standard deviation $\sigma = \sigma(x,t)$ of the stochastic temperature were performed from equations (16), (17) and (22), (23), respectively. Leaving in matrix infinite series only the terms involving ψ^i with the degree $i\leqslant 2$, these equations can be converted to: non-stationary moments provided that

$$(\Delta t/2)|\psi_{\mathrm{m}}|\cdot\|H_*\| < 1:$$

$$\bar{T}_* = \frac{\Delta t}{2}\sum_{i\leqslant 2}(-1)^iE(\psi^i)\left(\frac{\Delta t}{2}H_*\right)^i\bar{R}_*^{-1}F_*,$$

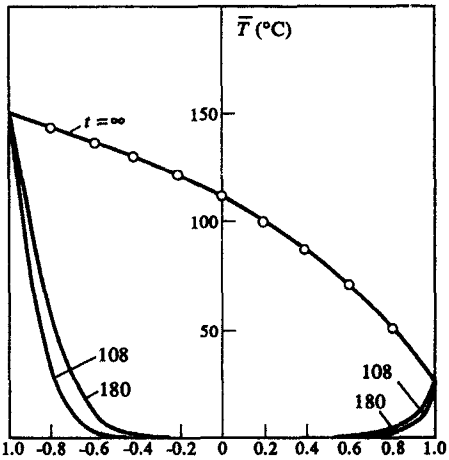

FIG. 1. The fields of the expected value for the stochastic temperature field of the plate at different times in the second-order approximation: (O)—exact solution.

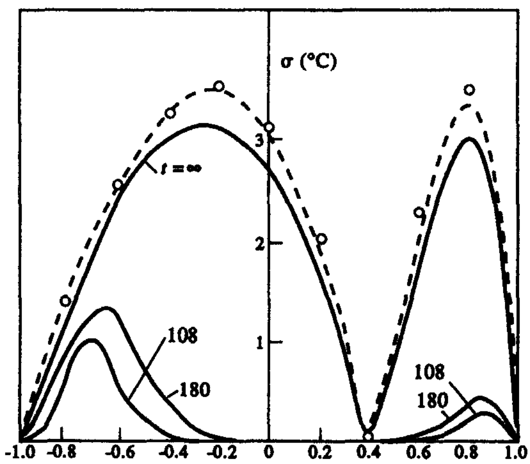

FIG. 2. The fields of the standard deviations in the stochastic temperature field of the plate at different times in the: (——) second-; and (– – –) fourth-order approximations: (O)—exact solution.

$$K_* = \frac{(\Delta t)^2}{4} \sum_{i+j \leq 2} (-1)^{i+j} E\{\psi^{i+j}\} \times \left(\frac{\Delta t}{2} H_*\right)^i \bar{R}_*^{-1} F_* F_*^{\mathrm{T}} (\bar{R}^{-1}) \left(\frac{\Delta t}{2} H_*^{-1}\right)^j;$$

stationary moments provided that $|\psi_{\mathrm{m}}| \cdot \|H\| < 1$:

$$\bar{T} = \sum_{i \leq 2} (-1)^i E\{\psi^i\} H^i \bar{R}^{-1} F,$$

$$K = \sum_{i+j \leq 2} (-1)^{i+j} E[\psi^{i+j}] H^i \bar{R}^{-1} F F^{\mathrm{T}} \bar{R}^{-1} (H^{\mathrm{T}})^j.$$

Note that in the given case the matrices H_* and H are deterministic and that all the odd moments of the random quantity $\psi(\omega)$ are equal to zero.

The results of calculations of non-stationary and stationary expectations and of standard deviations of the stochastic temperature field at the nodes of the plate grid for time $t = 108$, 180 s and $t = \infty$ are presented in Figs. 1 and 2. Also given in Fig. 2 is the stationary distribution of standard deviation calculated in a higher-order approximation (shown by dashed line in Fig. 2) when the terms ψ^i with the degree up to the fourth order inclusive are retained in the series, i.e. the summation in the expression for K is carried out for all i and j such that $i+j \leq 4$.

Comparison was also made between the approximate stationary moments $\bar{T}$ and σ and exact moments $\bar{T}_e(x) = E[T(x, \omega)]$ and $\sigma_e(x) = \{E[T(x, \omega) - T_e(x)]^2\}^{1/2}$ (shown by circles in Figs. 1 and 2), where:

$$T(x, \omega) = T_1 + (T_2 - T_1)\varphi(x, \omega)/\varphi(d, \omega),$$

$$\varphi(x, \omega) = \int_{-d}^{x} \mathrm{d}x/k(x, \omega).$$

Comparison of approximate stationary and exact moments shows that in the second-order approximation the expected value of $\bar{T}$ fully coincides with the exact one, whereas the standard deviation σ differs from the exact one by no more than 15%. The standard deviation σ, calculated in the fourth-order approximation, differs from the exact one by no more than 3%.

The dynamics of the standard deviation show that with time the maximum values of σ grow and shift to their stationary maximum values. A sharp decrease in σ at the point with the coordinate $x/d = 0.4$ is explained by the fact that different realizations of the stochastic temperature $T(x, \omega)$ intersect in the neighborhood of the point $x/d = 0.4$. Therefore, its statistical variation at this point is insignificant.

8. CONCLUSION

The proposed numerical method allows one to calculate non-stationary and stationary fields of expected values, correlation matrix and dispersion of the stochastic temperature field in an arbitrarily shaped three-dimensional body. The coefficients in the stochastic mathematical model are arbitrary random functions of coordinates and time with limited realizations. The discrete analog of the stochastic mathematical model is obtained by the methods of finite differences or finite elements. The moments of the stochastic temperature vector are obtained in the form of converging matrix series. Calculation of the moments with an accuracy sufficient for practical purposes requires a small number of series terms. The method can be easily programmed.

If random functions in the operator of the mathematical model are stationary random processes or Gaussian random functions or if they are statistically independent at all the modes of the grid in space and time, then the expressions for non-stationary and stationary moments simplify significantly.

Note that application of the method of finite elements for stochastic mathematical models is less convenient, in our opinion, as compared with the method of finite differences. This is due to the fact that to obtain the dispersion of the stochastic temperature distribution

$$D(x, t) = \sum_{i,j=1}^{n} E(T_i T_j)\varphi_i(x)\varphi_j(x)$$

[see equation (8)] it is necessary to calculate the full correlation matrix $K = E(T_i - T_j)$.

The assumptions about Gaussian white noises were not made, since these, as a rule, are not adequate to real random processes in heat conduction.

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